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A table of integrals generated by using an automatic symbolic integrator is presented. Included are indefinite integrals containing one or more Bessel functions, Laguerre functions. Legendre functions, or Hermite functions. Many of the results were not known prior to the development of the symbolic integration method used here.

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Message from the Chair

The record heat wave in Japan did not dim the spirits or the technical quality of the ISSAC'90 conference which was held in Nihon University, Tokyo (August 20 - 24). The 188 participants come from many different countries including a good number from the USSR. The conference also enjoyed strong local support in terms of attendance and industrial contribution. In parallel with the technical presentations were 10 demos on various workstations through the entire meeting. It is worth noting that ISSAC'90 marks several FIRST's: first time the Symposium ever comes to the Far East, first use of the ISSAC logo, and the first issue of ISSAC Proceedings to be distributed by Addison-Wesley.

Furthermore, a limited number of copies are available at ACM member discount: $18.00 for members (ACM order number 505900, ACM ISBN 0-89 791-401-5). Send order with payment to ACM Order Dept., P.O. Box 64145 Baltimore, MD 21264; credit card orders (1-800-342-6626). Non members, please order from Addison-Wesley Publishing Company, Order Dept, Jacob Way, Reading, MA, 01867; 1-800-447-2226 (A-W ISBN 0-201-54892-5).

Now that the ISSAC'90 is behind us we can look forward to the next meetings. ISSAC'91 (Bonn, Germany!) has announced a paper of deadline of Dec. 31, 1990. Let me encourage everyone to prepare and submit your contributions early. The proceedings will again be distributed by Addison-Wesley.

The decision was made in Tokyo that University of California at Berkeley will host ISSAC'92. Richard Fateman volunteered as Local Arrangements Chair. The organization of the conference and program committees is underway. A primary focus of ISSAC'92 will be the interface/integration of symbolic and numeric computation. There really should not be artificial boundaries between symbolic and numeric computations. Instead we should think of mathematical computation for science and engineering as a whole. This view is gaining support among the SIGSAM and SIGNUM membership. In fact, the two SIG's plan to join forces at ISSAC'92. If you are interested in helping with this conference, please feel free to contact me or others responsible for ISSAC'92.

Preparations for the election of new SIGSAM officers are in high gear. The nominating committee consists of R.J. Fateman (chair), A.C. Hearn, R. Jenks, and myself. We are making good progress in contacting people and getting commitments to run for office. Hopefully, statements from the candidates can also appear in the January issue of the Bulletin.

Table of Special Function Integrals

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This paper is the result of a suggestion by B. F. Caviness [1] to publish a table of integrals obtained using the special function integration technique that has been described in detail elsewhere [2]-[8], since only a much abbreviated version of the table could be accommodated in [7]. This integration technique permits symbolic evaluation of integrals of the form

$$I = \int f(x) \prod_{i=1}^{m} R_{\mu_i}^{(i)}(x)\, dx \,, \tag{1}$$

where f(x) is an essentially arbitrary function and ΠR is a product of any of a very broad class of special functions including, but not limited to, Bessel functions, Legendre functions, Hermite functions, Laguerre functions, Chebychev polynomials, Jacobi polynomials, and Gegenbauer polynomials. In [7] is described a method which can be used to enhance the on-board integrator of a symbolic software system to include analytical symbolic evaluation of integrals of this type. The method is currently being evaluated for possible inclusion in the program <u>Mathematica</u>™ [9].

The table is devided into two sections. In Sec. I are presented results derivable directly from the technique described in [7]. In Sec. II are presented results that were derived using the methods of [2]-[8], but these examples required human intervention for their evaluation. Current research in the area of this integration technique falls into two broad categories that roughly correspond to the subdivision of the tables. The first area corresponds to looking for heuristics beyond those presented in [7] for obtaining particular solutions to the ordinary, linear, inhomogeneous differential equation produced by the method. (Any particular solution of this differential equation is sufficient to produce an analytical expression for the associated special function integral.) The second area is associated with looking for algorithms that can eliminate the need for human intervention to produce results of the type presented in Sec. II of the tables. Current efforts center around attempts to automatically incorporate the use of generalized hypergeometric functions, ${}_pF_q$, or Meijer's G-function, into the particular solution search for these cases.

In the tables, Z and $\bar{Z}$ denote Bessel functions J or Y, P denotes Legendre functions of either the first or second kind, H denotes Hermite function, and L denotes Laguerre function. The parameters μ and ν are arbitrary numbers.

I. Results obtainable by fully automatic methods.

$$\int \frac{\sin(x) Z_\nu(x)\, dx}{x^{3/2}} = \frac{2[(2\nu+1)\ \sin(x) - 2x\ \cos(x)]}{x^{1/2}\ (2\nu-1)(2\nu+1)}\ Z_\nu(x) - \frac{4x^{1/2}\ \sin(x) Z_{\nu+1}(x)}{(2\nu-1)(2\nu+1)} \tag{1}$$

$$\int \frac{\cos(x) Z_\nu(x)\ dx}{x^{3/2}} = \frac{2[(2\nu+1)\ \cos(x) + 2x\ \sin(x)] Z_\nu(x)}{x^{1/2}\ (2\nu-1)(2\nu+1)} - \frac{4x^{1/2}\ \cos(x) Z_{\nu+1}(x)}{(2\nu+1)(2\nu-1)} \tag{2}$$

$$\int P_\nu(x)\ dx = -\frac{x}{\nu} P_\nu + \frac{1}{\nu} P_{\nu+1} \tag{3}$$

$$\int \ell n(1\pm x) P_\nu(x)\ dx = \left[-\frac{x}{\nu}\ \ell n(1\pm x) \pm \frac{1}{\nu(\nu+1)} - \frac{x}{\nu^2}\right] P_\nu + \left[\frac{1}{\nu}\ \ell n(1\pm x) + \frac{1}{\nu^2(\nu+1)}\right] P_{\nu+1} \tag{4}$$

$$\int e^{-x^2}\ H_\nu(x)\ dx = -e^{-x^2}\ H_{\nu-1}(x) \tag{5}$$

$$\int H_\nu(x) x^{-(\nu+3)}\ dx = \left[\frac{2x^{-\nu}}{(\nu+1)(\nu+2)} - \frac{x^{-(\nu+2)}}{(\nu+2)}\right] H_\nu(x) - \frac{2\nu}{(\nu+1)(\nu+2)}\ x^{-(\nu+1)} H_{\nu-1}(x) \tag{6}$$

$$\int x H_\nu(x)\ dx = \left(\frac{1+2x^2}{2(\nu+2)}\right) H_\nu(x) - \frac{\nu x}{(\nu+2)}\ H_{\nu-1}(x) \tag{7}$$

$$\int x e^{-(\nu+1)x} L_\nu(x)\ dx = \frac{e^{-(\nu+1)x}}{(\nu+1)}\ [-(1+x) L_\nu(x) + L_{\nu-1}(x)] \tag{8}$$

$$\int x(1+x)^{-(\nu+3)}L_\nu(x)\ dx$$

$$= \frac{(1+x)^{-(\nu+1)}}{(\nu+2)}\left[\left(\frac{x-\nu}{\nu+1} - \frac{x}{1+x}\right)L_\nu(x) + \left(\frac{\nu}{\nu+1}\right)L_{\nu-1}(x)\right] \tag{9}$$

$$\int \frac{Z_\mu(x)\bar{Z}_\nu(x)}{x}\, dx = \frac{1}{\mu+\nu}\left[Z_\mu\bar{Z}_\nu + \frac{xZ_\mu\bar{Z}_{\nu+1}}{(\mu-\nu)} - \frac{xZ_{\mu+1}\bar{Z}_\nu}{(\mu-\nu)}\right] \tag{10}$$

$$\int \frac{Z_\mu(x)\bar{Z}_\nu(x)\,dx}{x^2} = \frac{-(1+\mu+\nu+2\mu\nu+\mu\nu^2+\mu^2\nu-\mu^2-\mu^3-\nu^2-\nu^3+2x^2)Z_\mu\bar{Z}_\nu}{x(-1+\mu-\nu)(-1+\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)}$$

$$+ \frac{Z_\mu\bar{Z}_{\nu+1}}{(1-\mu-\nu)(1-\mu+\nu)} + \frac{Z_{\mu+1}\bar{Z}_\nu}{(1-\mu-\nu)(1+\mu-\nu)}$$

$$- \frac{2x\ Z_{\mu+1}\ \bar{Z}_{\nu+1}}{(1-\mu-\nu)(1-\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)} \tag{11}$$

$$\int \frac{Z_\mu(x)\bar{Z}_\nu(x)\,dx}{x^3} = \frac{-(2+\mu+\nu)(4\mu+4\nu+\mu\nu^2+\mu^2\nu-\mu^3-\nu^3+4x^2)}{x^2(\mu+\nu)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}\ Z_\mu\bar{Z}_\nu$$

$$- \frac{(4\mu\nu^2+2\mu^2\nu^2-4\mu^2-4\nu^3-\mu^4+4\nu^2-\nu^4+8x^2)}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}\ Z_\mu\bar{Z}_{\nu+1}$$

$$+ \frac{4\mu^2\nu+2\mu^2\nu^2+4\mu^2-\mu^4-4\nu^2-4\nu^3-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}\ Z_{\mu+1}\bar{Z}_\nu$$

$$- \frac{4}{[-\nu^2+(2-\mu)^2][-\nu^2+(2+\mu)^2]}\ Z_{\mu+1}\bar{Z}_{\nu+1} \tag{12}$$

$$\int \frac{Z_\nu^2(x)}{x^2}\, dx = \frac{1+2\nu+2x^2}{(4\nu^2-1)x}\ Z_\nu^2(x) - \frac{2}{-1+2\nu}\ Z_\nu(x)Z_{\nu+1}(x)$$

$$- \frac{2x}{1-4\nu^2}\ Z_{\nu+1}^2(x) \tag{13}$$

$$\int \frac{Z_\nu^2(x)}{x^4}\,dx = \frac{[-9-6\nu+x^2(6+16\nu+8\nu^2)+36\nu^2+24\nu^3+16x^4]}{3x^3(1-4\nu^2)(9-4\nu^2)}\, Z_\nu^2(x)$$

$$- \left[\frac{2(-3+4\nu+4\nu^2+8x^2)}{3x^2(1-2\nu)(9-4\nu^2)}\right] Z_\nu(x) Z_{\nu+1}(x)$$

$$- \frac{2(1-4\nu^2-8x^2)}{3x(1-4\nu^2)(9-4\nu^2)}\, Z_{\nu+1}^2(x) \qquad (14)$$

$$\int x^2\, Z_{1/3}^3(x)\,dx = \left(-\frac{4}{9}x - \frac{16}{81x}\right) Z_{1/3}^3(x)$$

$$- (4x/3) Z_{1/3}(x) Z_{4/3}^2(x)$$

$$+ \left(\frac{8}{9} + x^2\right) Z_{1/3}^2(x) Z_{4/3}(x) + \frac{2}{3}x^2\, Z_{4/3}^3(x) \qquad (15)$$

$$\int \frac{Z_1^4(x)}{x}\,dx = \frac{x^2}{4} Z_2^4(x) + \left(\frac{3}{4} + \frac{x^2}{4}\right) Z_1^4(x)$$

$$- \frac{3x}{2} Z_1(x) Z_2^3(x)$$

$$+ 6\left(\frac{1}{2} + \frac{x^2}{12}\right) Z_1^2(x)\; Z_2^2(x)$$

$$+ 4\left(-\frac{3x}{8} - \frac{1}{2x}\right) Z_1^3(x) Z_2(x) \qquad (16)$$

$$\int \frac{Z_3^4(x)}{x^3}\,dx = \left(\frac{1}{24} + \frac{1}{2x^2} + \frac{2}{x^4} + \frac{x^2}{378}\right) Z_3^4(x)$$

$$+ \left(\frac{5}{216} + \frac{2}{27x^2} + \frac{x^2}{378}\right) Z_4^4(x)$$

$$+ 4\left(-\frac{x}{108} - \frac{5}{54x} - \frac{1}{3x^3}\right) Z_3(x) Z_4^3(x)$$

$$+ 6\left(\frac{7}{216} + \frac{1}{3x^2} + \frac{4}{3x^4} + \frac{x^2}{1134}\right) Z_3^2(x) Z_4^2(x)$$

$$+ 4\left(-\frac{x}{108} - \frac{1}{8x} - \frac{1}{x^3} - \frac{4}{x^5}\right) Z_3^3(x) Z_4(x) \qquad (17)$$

$$\int P_\mu(x)\bar{P}_\nu(x)\,dx = \frac{-x\ P_\mu(x)\bar{P}_\nu(x)}{1+\mu+\nu}$$

$$- \frac{(1+\nu)}{(\mu-\nu)(1+\mu+\nu)}\ P_\mu(x)\bar{P}_{\nu+1}(x)$$

$$+ \frac{(1+\mu)}{(\mu-\nu)(1+\mu+\nu)}\ P_{\mu+1}(x)\bar{P}_\nu(x) \qquad (18)$$

$$\int x\ [P_\nu(x)]^2\ dx = \frac{-(1+\nu)}{2\nu}\left[\left(\frac{x^2+\nu}{1+\nu}\right)\ [P_\nu(x)]^2\right.$$

$$\left. - 2x\ P_\nu(x)P_{\nu+1}(x)\ +\ [P_{\nu+1}(x)]^2\right] \qquad (19)$$

$$\int \left[P_{1/2}(x)\right]^2 dx = \frac{9}{2}x\left[P_{3/2}(x)\right]^2 + \frac{x(-7+16x^2)}{2}\left[P_{1/2}(x)\right]^2$$

$$- 3(-1+2x)(1+2x)\ P_{1/2}(x)\ P_{3/2}(x) \qquad (20)$$

$$\int \left[P_{3/2}(x)\right]^2 dx = \frac{x(93-480x^2+512x^4)}{18}\left[P_{3/2}(x)\right]^2$$

$$+ \frac{25x(-3+8x^2)}{18}\left[P_{5/2}(x)\right]^2$$

$$- \frac{5(3-42x^2+64x^4)}{9}\ P_{3/2}(x)\ P_{5/2}(x) \qquad (21)$$

$$\int x^5\ P_{1/3}(x)P_{2/3}(x)\ dx = \left(-\frac{335}{2352} - \frac{515x^2}{392} + \frac{235x^4}{336} + \frac{x^6}{3}\right) P_{1/3}(x)P_{2/3}(x)$$

$$+ \left(\frac{685x}{784} - \frac{575x^3}{1176} - \frac{5x^5}{48}\right) P_{1/3}(x)P_{5/3}(x)$$

$$+ \left(\frac{235x}{196} - \frac{295x^3}{588} - \frac{2x^5}{21}\right) P_{2/3}(x)P_{4/3}(x)$$

$$+ \left(-\frac{5}{6} + \frac{65x^2}{196} + \frac{25x^4}{588}\right) P_{4/3}(x)P_{5/3}(x) \qquad (22)$$

$$\int x\ [P_{1/3}(x)]^3 dx = \left(\frac{125x^4}{12} - \frac{14}{3}x^2 - \frac{5}{12}\right) [P_{1/3}(x)]^3$$
$$+ (-4+20x^2)\ P_{1/3}(x)[P_{4/3}(x)]^2$$
$$+ (9x-25x^3)\ [P_{1/3}(x)]^2 P_{4/3}(x)$$
$$- \frac{16}{3}x\ [P_{4/3}(x)]^3 \tag{23}$$

$$\int x\ [P_{1/2}(x)]^4\ dx = \left(-\frac{5}{16} - \frac{19}{4}x^2\right) [P_{1/2}(x)]^4$$
$$+ \frac{81}{4}x\ P_{1/2}(x)[P_{3/2}(x)]^3$$
$$+ 6\left(-\frac{9}{16} - \frac{9}{2}x^2\right) [P_{1/2}(x)]^2[P_{3/2}(x)]^2$$
$$+ 4\left(\frac{33x}{16} + 3x^3\right) [P_{1/2}(x)]^3 P_{3/2}(x)$$
$$- \frac{81}{16} [P_{3/2}(x)]^4 \tag{24}$$

$$\int e^{-x^2}\ \bar{H}_\mu(x)\ \bar{H}_\nu(x)dx = \frac{e^{-x^2}}{2(\mu-\nu)} \left[-\bar{H}_\mu(x)\ \bar{H}_{\nu+1}(x) + \bar{H}_{\mu+1}(x)\ \bar{H}_\nu(x)\right] \tag{25}$$

$$\int xe^{-x^2}\ \bar{H}_\mu(x)\ \bar{H}_\nu(x)\ dx = \frac{e^{-x^2}}{2} \left[-\frac{\bar{H}_\mu(x)\ \bar{H}_\nu(x)(1+\mu+\nu)}{(1-\mu+\nu)(1+\mu-\nu)}\right.$$
$$+ \bar{H}_{\mu+1}(x)\ \bar{H}_\nu(x)\ \frac{x}{(1+\mu-\nu)} + \bar{H}_\mu(x)\ \bar{H}_{\nu+1}(x)\ \frac{x}{(1-\mu+\nu)}$$
$$\left. - \frac{\bar{H}_{\mu+1}(x)\ \bar{H}_{\nu+1}(x)}{(1-\mu+\nu)(1+\mu-\nu)}\right] \tag{26}$$

$$\int x^2 e^{-x^2} \bar{H}_\mu(x)\ \bar{H}_\nu(x)\ dx = e^{-x^2} \left[\frac{-\bar{H}_\mu(x)\ \bar{H}_\nu(x)\ x(\mu+\nu)}{(2-\mu+\nu)(2+\mu-\nu)}\right]$$
$$+ \bar{H}_{\mu+1}(x)\ \bar{H}_\nu(x)\ \frac{(2+\mu+3\nu+2\mu x^2-2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2)}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$- \bar{H}_\mu(x)\ \bar{H}_{\nu+1}(x)\ \frac{(2+3\mu+\nu-2\mu x^2+2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2)}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$- \bar{H}_{\mu+1}(x)\ \bar{H}_{\nu+1}(x)\ \frac{x}{(2-\mu+\nu)(2+\mu-\nu)} \tag{27}$$

$$\int e^{-3x^2} x^2 H^3_{2/3}(x)\ dx = e^{-3x^2} \left\{ -\frac{1}{12} x\ (5+6x^2)\ H^3_{2/3}(x) \right.$$

$$+ \frac{1}{8}\ (1+6x^2)\ H^2_{2/3}(x) H_{5/3}(x)$$

$$- \frac{3}{8}\ x\ H_{2/3}(x) H^2_{5/3}(x)$$

$$\left. + \frac{1}{16}\ H^3_{5/3}(x) \right\} \tag{28}$$

$$\int e^{-x}\ L_\mu(x)\ \bar{L}_\nu(x)\ dx = e^{-x}\Big[L_\mu(x)\ \bar{L}_\nu(x) + \frac{(1+\nu)}{(\mu-\nu)}\ L_\mu(x)\ \bar{L}_{\nu+1}(x)$$

$$- \frac{(1+\mu)}{(\mu-\nu)}\ L_{\mu+1}(x)\ \bar{L}_\nu(x)\Big] \tag{29}$$

$$\int x e^{-x}\ L_\mu(x)\ \bar{L}_\nu(x)\ dx = e^{-x}\left[-\frac{(1+\mu+\nu-x+2\mu\nu+\mu^2 x+\nu^2 x-2\mu\nu x)}{(1-\mu+\nu)(1+\mu-\nu)}\ L_\mu(x)\ \bar{L}_\nu(x) \right.$$

$$+ \frac{(1+\nu)(1+2\mu-\mu x+\nu x)}{(\mu-\nu)(1-\mu+\nu)}\ L_\mu(x)\ \bar{L}_{\nu+1}(x)$$

$$- \frac{(1+\mu)(1+2\nu+\mu x-\nu x)}{(\mu-\nu)(1+\mu-\nu)}\ L_{\mu+1}(x)\ \bar{L}_\nu(x)$$

$$\left. - \frac{2(1+\mu)(1+\nu)}{(1-\mu+\nu)(1+\mu-\nu)}\ L_{\mu+1}(x)\ \bar{L}_{\nu+1}(x) \right] \tag{30}$$

$$\int e^{-3x}\ x\ L^3_{2/3}(x)\ dx = e^{-3x} \left\{ L^3_{2/3}(x)\left[\frac{125}{24} - \frac{625x}{24} \right.\right.$$

$$\left. + \frac{853x^2}{16} - \frac{675x^3}{16} + \frac{225x^4}{16} - \frac{27x^5}{16} \right]$$

$$+ L^3_{5/3}(x)\ \left[-\frac{125}{24} + \frac{125x}{12} - \frac{125x^2}{16} \right]$$

$$+ 3\ \left[\frac{125}{24} - \frac{125x}{8} + \frac{275x^2}{16} - \frac{75x^3}{16} \right] L_{2/3}(x) L^2_{5/3}(x)$$

$$\left. + 3\ \left[-\frac{125}{24} + \frac{125x}{6} - \frac{515x^2}{16} + \frac{135x^3}{8} - \frac{45x^4}{16} \right] L^2_{2/3}(x) L_{5/3}(x) \right\} \tag{31}$$

II. Results requiring human intervention. Here, ℓ denotes integer.

$$\int x^{\ell} Z_\mu(x)\ \bar{Z}_\nu(x)\ dx = A_{00}(x)\ Z_\mu(x)\ \bar{Z}_\nu(x) + A_{01}(x)\ Z_\mu(x)\ \bar{Z}_{\nu+1}(x)$$

$$+ A_{10}(x)\ Z_{\mu+1}(x)\ \bar{Z}_\nu(x) + A_{11}(x)\ Z_{\mu+1}(x)\ \bar{Z}_{\nu+1}(x)\ , \qquad (32)$$

where

$$A_{00} = \frac{x}{2(\mu+\nu)} D^3A_{11} + \frac{(3+\mu+\nu)}{2(\mu+\nu)} D^2A_{11}$$

$$+ \frac{(-7-3\mu-3\nu-2\mu\nu+\mu^2+\nu^2-4x^2)}{2x(\mu+\nu)} DA_{11}$$

$$+ \frac{(-2-\mu-\nu)(-4-2\mu\nu+\mu^2+\nu^2-2x^2)}{2x^2(\mu+\nu)} A_{11} + \frac{x^{\ell+1}}{\mu+\nu}\ ,$$

$$A_{01} = \frac{-x^2}{2(\mu^2-\nu^2)} D^3A_{11} + \frac{3x}{2(\mu^2-\nu^2)} D^2A_{11}$$

$$- \frac{(7-3\mu^2-\nu^2+4x^2)}{2(\mu^2-\nu^2)} DA_{11} + \frac{(4+\mu\nu^2-3\mu^2-\mu^3-\nu^2+2x^2)}{x(\mu^2-\nu^2)} A_{11} + \frac{x^{\ell+2}}{(\mu^2-\nu^2)}\ ,$$

$$A_{10} = \frac{x^2}{2(\mu^2-\nu^2)} D^3A_{11} - \frac{3x}{2(\mu^2-\nu^2)} D^2A_{11} + \frac{(7-\mu^2-3\nu^2+4x^2)}{2(\mu^2-\nu^2)} DA_{11}$$

$$- \frac{(4+\mu^2\nu-\mu^2-3\nu^2-\nu^3+2x^2)}{x(\mu^2-\nu^2)} A_{11} - \frac{x^{\ell+2}}{(\mu^2-\nu^2)}\ ,$$

and

$$A_{11}(x) = x^{\ell+3} \sum_{n=0}^{n'} d_n x^{2n}\ ,$$

where

$$d_0 = \frac{2(\ell+1)}{(\ell+3)^4-8(\ell+3)^3+2(12-\mu^2-\nu^2)(\ell+3)^2-8(\ell+3)(4-\mu^2-\nu^2)+[(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2]}\ ,$$

$$\{(3+2n+\ell)^4-8(3+2n+\ell)^3+2(12-\mu^2-\nu^2)(3+2n+\ell)^2$$

$$-8(3+2n+\ell)(4-\mu^2-\nu^2)+[(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2]\}\ d_n$$

$$= -4(1+2n+\ell)(2n+\ell)d_{n-1} \qquad\qquad n' \geq n > 0 \quad ,$$

and $d_n = 0$ if $n > n'$. The parameter n' is given by

$$n' = \begin{cases} 0 \text{ if } \ell = -1 \\ \frac{|\ell|}{2} - 1 \text{ if } \ell < -1 \text{ and even,} \\ \frac{|\ell|-3}{2} \text{ if } \ell < -1 \text{ and odd,} \\ \infty \text{ if } \ell \geq 0 \end{cases}$$

If $\ell \geq 0$, d_n in the sum for A_{11} can be written in the form

$$d_n = \frac{2(-4)^n(\ell+1)_{2n+1}}{\prod_{k=0}^{n} g(k,\ell,\mu,\nu)} \quad ,$$

where $(\ell+1)_{2n+1}$ is a shifted factorial, $(\ell+1)_{2n+1} \equiv (\ell+1)(\ell+2)\ldots(2n+\ell+1)$, and g is defined by

$$g(k,\ell,\mu,\nu) \equiv (2k+\ell+3)^4-8(2k+\ell+3)^3+2(12-\mu^2-\nu^2)(2k+\ell+3)^2$$

$$-\ 8(2k+\ell+3)(4-\mu^2-\nu^2)+[(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2] \ .$$

$$\int x^\ell Z_\nu{}^2(x)\ dx = A_{00}(x)\ Z_\nu{}^2(x) + 2A_{01}(x)\ Z_\nu(x)Z_{\nu+1}(x) + A_{11}(x)\ Z_{\nu+1}^2(x), \quad (33)$$

where

$$A_{00}(x) = \frac{1}{2} D^2 A_{11}(x) - \frac{(3+2\nu)}{2x} DA_{11}(x)$$

$$+ \frac{(2+2\nu+x^2)}{x^2} A_{11}(x)$$

$$A_{01}(x) = \frac{1}{2} DA_{11}(x) - \frac{(1+\nu)}{x} A_{11}(x) \ ,$$

and

$$A_{11}(x) = x\ y(x)$$

$$y = \sum_{n=0}^{\frac{\ell-1}{2}} b_n x^{2n+1} ,$$

$$b_{\frac{\ell-1}{2}} = 1/2\ell$$

$$b_n = \frac{2(n+1)[\nu^2-(n+1)^2]b_{n+1}}{2n+1} ,$$

if $0 \le n \le \frac{\ell-3}{2}$ and $\ell \ge 3$, but positive odd integer only.

Also,

$$y = \frac{-x^\ell}{2\ell} \; {}_4F_1\left[\frac{1-\ell}{2}, \frac{-\ell-2\nu+1}{2}, \frac{-\ell+2\nu+1}{2}, 1; \frac{-\ell+3}{2}; -\frac{1}{x^2}\right] ,$$

if ℓ=negative odd integer, ν=integer but $\neq 0$,

and

$$y = \frac{2x^{\ell+2}}{(\ell+1)[(\ell+1)^2-4\nu^2]} \; {}_2F_3\left(\frac{\ell+2}{2}, 1; \frac{\ell+3}{2}, \frac{\ell-2\nu+3}{2}, \frac{\ell+2\nu+3}{2}; -x^2\right),$$

if ℓ=0 or even integer (positive or negative) but $\ell\pm2\nu\neq$negative odd integer.

$$\int x^\ell P_\mu(x)P_\nu(x)\,dx = A_{00}(x)\; P_\mu(x)P_\nu(x) + A_{01}\; P_\mu(x)P_{\nu+1}(x) + A_{10}\; P_{\mu+1}(x)P_\nu(x) + A_{11}\; P_{\mu+1}(x)P_{\nu+1}(x) , \tag{34}$$

where

$$A_{00}(x) = \frac{-x}{1+\mu+\nu} f(x) - \frac{(x^2-1)}{(\mu-\nu)(1+\mu+\nu)} Df(x) + g_{000}(x)\; A_{11}(x) + g_{001}(x)\; DA_{11}(x) + g_{002}(x)\; D^2A_{11}(x) + g_{003}(x)\; D^3A_{11}(x) + g_{004}(x)\; D^4A_{11}(x)$$

$$A_{01}(x) = \frac{-(1+\nu)}{(\mu-\nu)(1+\mu+\nu)} f(x) + g_{010}(x)\, A_{11}(x) + g_{011}(x)\, DA_{11}(x)$$

$$+ g_{012}(x)\, D^2A_{11}(x) + g_{013}(x)\, D^3A_{11}(x)$$

$$A_{10}(x) = \frac{(1+\mu)}{(\mu-\nu)(1+\mu+\nu)} f(x) + g_{100}(x)\, A_{11}(x) + g_{101}(x)\, DA_{11}(x)$$

$$+ g_{102}(x)\, D^2A_{11}(x) + g_{103}(x)\, D^3A_{11}(x) .$$

The parametric functions g are given by

$$g_{000}(x) = \frac{\begin{aligned}[\mu+\nu+\mu\nu+\mu\nu^2-2\nu x^2+\mu^2\nu+\mu^2\nu^2 \\ - \nu^2x^2+2\nu^3x^2+\nu^4x^2-\mu\nu x^2-4\mu\nu^2x^2 \\ - \mu\nu^3x^2+2\mu^2\nu x^2-\mu^2\nu^2x^2+\mu^3\nu x^2 \\ + \mu^2-2\nu^3-\nu^4]\end{aligned}}{(1+\mu)(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{001}(x) = \frac{\begin{aligned}-x[12-12\nu-3\mu\nu^2-4\mu x^2+16\nu x^2+3\mu^2\nu \\ + 3\mu^2x^2+\mu^3x^2+9\nu^2x^2-\nu^3x^2 \\ + 3\mu\nu^2x^2-3\mu^2\nu x^2-3\mu^2-\mu^3-9\nu^2 \\ + \nu^3-12x^2]\end{aligned}}{2(1+\mu)(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{002}(x) = \frac{\begin{aligned}-[(x^2-1)(8-\mu-3\nu-5\mu x^2+9\nu x^2+\mu^2x^2 \\ + 3\nu^2x^2-\mu^2-3\nu^2-24x^2]\end{aligned}}{2(1+\mu)(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{003}(x) = \frac{x(-1+x^2)^2\ (10+\mu-\nu)}{2(1+\mu)(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{004}(x) = \frac{(-1+x^2)^3}{2(1+\mu)(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{010}(x) = \frac{-\ \mu x(1+\mu-\nu)(2+\mu+\nu)}{(1+\mu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{011}(x) = -[2-3\mu-\nu+3\mu x^2+\nu x^2+3\mu^2x^2 \frac{+\ \nu^2x^2-3\mu^2-\nu^2-6x^2]}{2(1+\mu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{012}(x) = \frac{3x(-1+x^2)}{(1+\mu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{013}(x) = \frac{(-1+x^2)^2}{2(1+\mu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{100}(x) = \frac{\nu x(1-\mu+\nu)(2+\mu+\nu)}{(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{101}(x) = [2-\mu-3\nu+\mu x^2+3\nu x^2+\mu^2x^2 \frac{+\ 3\nu^2x^2-\mu^2-3\nu^2-6x^2]}{2(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{102}(x) = \frac{-3x(-1+x^2)}{(1+\nu)(\mu-\nu)(1+\mu+\nu)}$$

$$g_{103}(x) = \frac{-(-1+x^2)^2}{2(1+\nu)(\mu-\nu)(1+\mu+\nu)} ,$$

and,

$$A_{11}(x) = \sum_{p}^{\ell-1}{}' \, b_p x^p .$$

The quantities b_p in the sum for A_{11} are constants given by

$$b_{\ell-1} = \frac{(2\ell)(1+\mu)(1+\nu)}{(\mu+\nu+\ell+1)(\mu+\nu-\ell+1)[(\mu-\nu)^2-\ell^2]}$$

$$b_p = -(p+1)(p+2)\{(p+3)(p+4)b_{p+4}(1-\delta_{p,\ell-3}) \frac{+\ 2b_{p+2}[\mu+\nu+\mu^2+\nu^2-(p+2)^2]\}}{(\mu+\nu+p+2)(\mu+\nu-p)[(\mu-\nu)^2-(p+1)^2]} .$$

$$(0 \leq p < \ell-1)$$

The prime on the summation for A_{11} signifies that p=0, 2, ..., ℓ-1 when ℓ is odd, and p=1, 3, ..., ℓ-1 when ℓ is even.

$$\int x^{\ell}[P_\nu(x)]^2\, dx = A_{00}(x)\ [P_\nu(x)]^2 + 2A_{01}(x)P_\nu(x)P_{\nu+1}(x)$$

$$+ A_{11}(x)\ [P_{\nu+1}(x)]^2 , \tag{35}$$

where

$$A_{00}(x) = \frac{(-1+x^2)^2}{2(1+\nu)^2} D^2A_{11}(x) + \frac{(-1+x^2)(2+\nu)}{(1+\nu)^2}\ x\ DA_{11}(x)$$

$$+ \frac{(\nu+x^2)}{(1+\nu)} A_{11}(x)$$

$$A_{01}(x) = \frac{-(-1+x^2)}{2(1+\nu)} DA_{11}(x) - xA_{11}(x)$$

$$A_{11}(x) = \sum_{n=0}^{\frac{\ell-1}{2}} a_{2n}\ x^{2n} \qquad (\ell = \text{positive odd integer}) ,$$

and

$$a_{\ell-1} = \frac{2(\nu+1)^2}{(\ell-1)(\ell-2)(\ell+3)-2(\ell-1)(2\nu+2\nu^2-3)-4\nu(\nu+1)} ,$$

$$a_{2n} = \frac{(2n+2)\{2[(2n+1)(2n+3)+(1-2\nu-2\nu^2)]a_{2n+2}+(2n+4)(2n+3)(\delta_{2n,\ell-3}-1)a_{2n+4}\}}{2n(2n-1)(2n+4)-4n(2\nu+2\nu^2-3)-4\nu(\nu+1)}$$

$$0 \le n < \frac{\ell-1}{2} ,\ \ell = \text{positive odd integer}$$

and a_{2n}=0 for n > (ℓ-1)/2. The index n may assume the value zero only if $\ell \neq 1$. If ℓ=1, the series for A_{11} reduces to a single term that is given by the expression for $a_{\ell-1}$ above.

If, ℓ=0, ν=one half odd integer, but $\neq$ -1/2, then

$$A_{11}(x) = \sum_{n=1}^{n'} b_n x^{2n-1}$$

where $n' \equiv \pm(2\nu+1)/2$, and

$$b_{n+2}(2n+1)(2n+2)(2n+3)-2(2n+1)[(1+2n)^2-2\nu(\nu+1)]b_{n+1}$$

$$-2n[(2\nu+1)^2-4n^2]b_n=2(\nu+1)^2\delta_{n,0}\ ; \quad n=0,\ 1,\ \ldots,\ \frac{\pm(2\nu+1)}{2}\ ,$$

$$\text{and } b_n=0 \text{ for } n > \frac{\pm(2\nu+1)}{2}\ .$$

The preceding expression represents $\frac{\pm(2\nu+1)}{2}$ simultaneous algebraic equations in the $\frac{\pm(2\nu+1)}{2}$ unknowns b_1, b_2, ..., $b_{\pm(2\nu+1)/2}$.

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